

Small oriented cycle double cover of graphs

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Abstract

A small oriented cycle double cover (SOCDC) of a bridgeless graph G on n vertices is a collection of at most $n - 1$ directed cycles of the symmetric orientation, G_s , of G such that each edge of G_s lies in exactly one of the cycles. It is conjectured that every 2-connected graph except two complete graphs K_4 and K_6 has an SOCDC. In this paper, we study graphs with SOCDC and prove that the minimal counterexample to this conjecture is a snark.

Keywords: Cycle double cover, Small cycle double cover, Oriented cycle double cover, Small oriented cycle double cover.

1 Introduction

We denote by G a finite undirected graph with vertex set V and edge set E with no loops or multiple edges. The **symmetric orientation** of G , denoted by G_s , is an oriented graph obtained from G by replacing each edge of G by a pair of opposite directed arcs. An **even graph** (**odd graph**) is a graph such that each vertex is incident to an even (odd) number of edges. A **directed even graph** is a graph such that for each vertex its out-degree equals to its in-degree. A **cycle** (**a directed cycle**) is a minimal non-empty even graph (directed even graph). We denote every directed cycle C and directed path P on n vertices with vertex set $\{v_1, \dots, v_n\}$ and directed edge set $E(C) = \{v_i v_{i+1}, v_n v_1 : 1 \leq i \leq n - 1\}$ and $E(P) = \{v_i v_{i+1} : 1 \leq i \leq n - 1\}$ by $C = [v_1, \dots, v_n]$, and $P = (v_1, \dots, v_n)$, respectively.

A **cycle double cover** (CDC) \mathcal{C} of a graph G is a collection of cycles in G such that every edge of G belongs to exactly two cycles of \mathcal{C} . Note that the cycles

are not necessarily distinct. It can be easily seen that a necessary condition for a graph to have a CDC is that the graph has no cut edge which is called a bridgeless graph. Seymour [17] in 1979 conjectured that every bridgeless graph has a CDC. No counterexample to the CDC conjecture is known. It is proved that the minimal counterexample to the CDC conjecture is a bridgeless cubic graph with edge chromatic number equal to 4, which is called a **snark**.

A **small cycle double cover (SCDC)** of a graph on n vertices is a CDC with at most $n - 1$ cycles. There exist simple graphs of order n for which any CDC requires at least $n - 1$ cycles (e.g., $K_n, n \geq 3$). Furthermore, no simple bridgeless graph of order n is known to require more than $n - 1$ cycles in a CDC. Note that clearly it is false if not restricted to simple graphs. Bondy [3] conjectured that every simple bridgeless graph has an SCDC. For more results on the CDC conjecture see [7, 19].

A **perfect path double cover (PPDC)** of a graph G is a collection \mathcal{P} of paths in G such that each edge of G belongs to exactly two members of \mathcal{P} and each vertex of G occurs exactly twice as an end of a path in \mathcal{P} [2]. In [11] it is proved that every simple graph has a PPDC.

The CDC conjecture has many stronger forms. In this paper, we consider the oriented version of these conjectures.

An **oriented cycle double cover (OCDC)** is a CDC in which every cycle can be oriented in such a way that every edge of the graph is covered by two directed cycles in two different directions.

Conjecture 1.1 [8] (Oriented CDC conjecture) *Every bridgeless graph has an OCDC.*

No counterexample to this conjecture is known. It is clear that the validity of the OCDC conjecture implies the validity of the CDC conjecture. While there is a CDC of the Petersen graph that can not be oriented in such a way that forms an OCDC.

Definition 1.2 *A small oriented cycle double cover (SOCDC) of a graph on n vertices is an OCDC with at most $n - 1$ directed cycles.*

An **oriented perfect path double cover (OPPDC)** of a graph G is a collection of directed paths in the symmetric orientation G_s such that each edge of G_s lies in exactly one of the paths and each vertex of G appears just once as a beginning and just once as end of a directed path. Maxová and Nešetřil in [14] showed that two complete graphs K_3 and K_5 have no OPPDC and in [13], they conjectured every connected graph except K_3 and K_5 has an OPPDC.

The **join** of two simple graphs G and H , $G \vee H$, is the graph obtained from the disjoint union of G and H by adding the edges $\{uv : u \in V(G), v \in V(H)\}$.

The existence of a PPDC for graphs in general is equivalent to the existence of an SCDC for the bridgeless graph obtained by joining a new vertex to all other vertices [2]. The following theorem denotes a relation between OPPDC and SOCDC.

Theorem 1.3 [14] *Let G be a connected graph. The graph G has an OPPDC if and only if $G \vee K_1$ has an SOCDC.*

In the following theorem a list of some families of graphs that admit an OPPDC is provided. Therefore by Theorem 1.3, the join of graphs satisfying at least one of the conditions in below and K_1 admit an *SOCDC*.

Theorem 1.4 [1, 14] *Let $G \neq K_3$ be a graph. In each of the following cases, G has an OPPDC.*

- (i) G is a union of two arbitrary trees.
- (ii) G is an odd graph.
- (iii) G has no adjacent vertices of degree greater than two.
- (iv) G is a 2-connected graph of order n and $|E(G)| \leq 2n - 1$.
- (v) $G = L(T)$, for some tree T .
- (vi) $G = L(H)$, where the degree of no adjacent vertices in H have the same parity.
- (vii) G is a graph with $\Delta(G) \leq 4$ and $\delta(G) \leq 3$.
- (viii) G is a separable 4-regular graph.

In what follows we have three sections. Section 2 deals with graphs with a small oriented cycle double cover. It is conjectured that every 2-connected graph except two complete graphs K_4 and K_6 has an SOCDC. In Section 3, we prove that the minimal counterexample to this conjecture is also a snark. Finally in Section 4, some more relations between OPPDC and SOCDC are given.

2 The small oriented cycle double cover

The natural question is that which simple bridgeless graphs of order n have an OCDC with at most $n - 1$ cycles (SOCDC)?

Since K_3 and K_5 have no OPPDC, by Theorem 1.3, K_4 and K_6 have no SOCDC. It is known that every K_{2n-1} , $n \geq 4$, has an OPPDC, thus by Theorem 1.3, every K_{2n} , $n \geq 4$, has an SOCDC. Moreover, every K_{2n+1} has an SOCDC, since K_{2n+1} has a Hamiltonian cycle decomposition [18].

The following proposition shows that if every block of a graph G has an SOCDC, then G has also an SOCDC.

Proposition 2.1 *If $G = G_1 \cup G_2$ and $V(G_1) \cap V(G_2) = \{v\}$ which G_i is a graph with an SOCDC, $i = 1, 2$; then G also has an SOCDC.*

Proof. Let G_i be a graph of order n_i with an SOCDC, say \mathcal{C}_i , $i = 1, 2$, and $V(G_1) \cap V(G_2) = \{v\}$. Therefore, $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$ is an OCDC of size at most $n_1 + n_2 - 2$ of G . Note that G is a graph of order $n_1 + n_2 - 1$. Thus, \mathcal{C} is an SOCDC of G . ■

Moreover, Proposition 2.1 directly concludes the following corollaries. A **block graph** is a graph for which each block is a clique.

Corollary 2.2 *Every block graph with no block of order 2, 4 and 6 has an SOCDC.*

Since the line graph of every tree is a block graph, the following result obtained which is an oriented version of existence of SCDC of line graph of trees [12].

Corollary 2.3 *If T is a tree without vertices of degree 2, 4 or 6, then $L(T)$ has an SOCDC.*

In the following proposition, we construct some graphs with no SOCDC. In fact, we show that the difference $|\mathcal{C}| - (n - 1)$ could be large enough for every OCDC, \mathcal{C} of some bridgeless graph of order n .

Let $V(K_4) = \{v_1, v_2, v_3, v_4\}$. The collection $\mathcal{C} = \{[v_1, v_2, v_4], [v_2, v_1, v_3], [v_3, v_4, v_2], [v_4, v_3, v_1]\}$ is an OCDC of K_4 . Since K_4 has six edges, if \mathcal{C} is an arbitrary OCDC of K_4 , then $|\mathcal{C}| \leq (2 \times 6)/3 = 4$. Thus, every OCDC of K_4 is of size 4.

Let $V(K_6) = \{v_1, \dots, v_6\}$. The collection $\mathcal{C} = \{[v_1, v_2, v_3, v_4, v_5, v_6], [v_2, v_6, v_3, v_5, v_4], [v_1, v_5, v_2, v_4, v_3], [v_1, v_4, v_6, v_2, v_5], [v_1, v_6, v_5, v_3, v_2], [v_1, v_3, v_6, v_4]\}$ is an OCDC of K_6 of size 6.

Proposition 2.4 *For every integer $r \geq 1$, there exists a bridgeless graph G of order n such that every OCDC of G has $(n - 1) + r$ directed cycles.*

Proof. Let P be a path of length r with $V(P) = \{v_1, \dots, v_{r+1}\}$ and $E(P) = \{v_i v_{i+1} : 1 \leq i \leq r\}$. Assume that G is a graph obtained from P by replacing each edge $v_i v_{i+1}$ of P with a clique K_4 , say K_4^i , where $V(K_4^i) = \{v_i, v'_i, v_{i+1}, v'_{i+1}\}$,

$1 \leq i \leq r$. Every OCDC of G is decomposable to r OCDC of K_4 . Moreover, every OCDC of K_4 has four cycles. Therefore, every OCDC of G has $4r$ cycles. Note that $|V(G)| = 3r + 1$, thus every OCDC of G has $(|V(G)| - 1) + r$ cycles. ■

Until now the known bridgeless graphs except K_4 and K_6 without SOCDC have cut vertex. Let look at the 2-connected graphs.

Let G be a 2-connected graph of order n with vertex cut $\{v_1, v_2\}$ and $G = G_1 \cup G_2$, where $V(G_1) \cap V(G_2) = \{v_1, v_2\}$ and $|V(G_i)| = n_i$, $i = 1, 2$. Assume that $G_i \cup \{v_1 v_2\}$ has an SOCDC, \mathcal{C}_i , $i = 1, 2$. Let $C_i^j, j = 1, 2$, be the two directed cycles in \mathcal{C}_i , $i = 1, 2$, which include the directed edge $v_j v_{j+1}$, where subscripts are reduced modulo 2. If $v_1 v_2 \in E(G)$, then we define

$$\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \{C_1^1 \Delta C_2^2\} \setminus \{C_1^1, C_2^2\}.$$

The collection \mathcal{C} is an OCDC of G , where

$$\begin{aligned} |\mathcal{C}| &= |\mathcal{C}_1| + |\mathcal{C}_2| - 1 \leq (n_1 - 1) + (n_2 - 1) - 1 \\ &\leq (n_1 + n_2) - 3 \\ &\leq (n + 2) - 3 = n - 1. \end{aligned}$$

If $v_1 v_2 \notin E(G)$, then we define

$$\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \{C_1^1 \Delta C_2^2, C_1^2 \Delta C_2^1\} \setminus \{C_1^1, C_1^2, C_2^1, C_2^2\}.$$

The collection \mathcal{C} is an OCDC of G , where $|\mathcal{C}| \leq n - 2$.

Furthermore, if $G_1 \cup \{v_1 v_2\} = K_4$ or K_6 , $v_1 v_2 \notin E(G)$, and $G_2 \cup \{v_1 v_2\}$ has an SOCDC, by the similar argument in above using the given SOCDC for K_4 and K_6 of size 4 and 6, an SOCDC for G is obtained.

This fact motivates us to present the following conjecture.

Conjecture 2.5 (SOCDC conjecture) *Every simple 2-connected graph except K_4 and K_6 admits an SOCDC.*

The above conjecture has a close relation to the following conjecture.

Conjecture 2.6 [4] (Hajós' conjecture) *If G is a simple, even graph of order n , then G can be decomposed into $\lfloor (n - 1)/2 \rfloor$ cycles.*

If the Hajós' conjecture holds, then every even graph has an SOCDC obtained by taking two copies of the cycles used in its decomposition, in two opposite directions.

An edge of a graph G is said to be contracted if it is deleted and its two ends are identified. A **minor** of G is a graph obtained from G by deletions of vertices, and

deletions and contractions of edges. The graph obtained from K_6 by deleting an edge is denoted K_6^- . A K_6^- -minor free graph is a graph that does not contain K_6^- as a minor.

As the Hajós' conjecture is true for even graphs with maximum degree four [5], planar graphs [16], projective graphs (a projective graph is a graph G which is embeddable on the projective plane.), and K_6^- -minor free graphs [4], these graphs have an SOCDC.

Proposition 2.7 *Let G be an even graph. In each of the following cases, G has an SOCDC.*

- (i) $\Delta(G) = 4$.
- (ii) G is planar.
- (iii) G is a projective graph.
- (iv) G is K_6^- -minor free.

Klimmek [9] proved that every even line graph of order n has a cycle decomposition into $\lfloor (n-1)/2 \rfloor$ cycles, thus the Hajós' conjecture holds for such graphs. Since a line graph, $L(G)$, is even if and only if every component of G is either even or odd, the line graph of every even graph and of every odd graph has an SOCDC.

Proposition 2.8 *If G is an even or an odd graph and has no cut vertex of degree 2, then $L(G)$ has an SOCDC.*

The following proposition considers another class of graphs with OCDC which also has SOCDC.

Proposition 2.9 *If G has an OCDC, \mathcal{C} , and the girth of G , $g(G)$, is greater than maximum degree, $\Delta(G)$, then \mathcal{C} is also an SOCDC of G .*

Proof. Let \mathcal{C} be an OCDC of G . Note that each edge of G is covered twice by elements of \mathcal{C} , therefore,

$$g(G)|\mathcal{C}| \leq \sum_{C \in \mathcal{C}} |E(C)| = 2|E(G)| = \sum_{v \in V(G)} d(v) \leq |V(G)|\Delta(G).$$

Since $g(G) > \Delta(G)$, we have $|\mathcal{C}| \leq |V(G)| - 1$. Hence, \mathcal{C} is an SOCDC of G . ■

In general case, if G has an OCDC and $|E(G)| < g(G)|V(G)|/2$, then G has an SOCDC. For example every triangle-free cubic graph that has an OCDC, has an SOCDC.

It can be proved that an OCDC for planar graphs can be obtained from their planar embedding and some planar graph has also SOCDC.

Theorem 2.10 *Every bridgeless planar graph has an OCDC.*

Proof. Let G be a bridgeless planar graph and consider a fixed planar embedding of G . We show that the edges of faces of G can be oriented in such a way that the collection of its faces forms an OCDC.

We proceed by induction on the number of faces of G . Obviously, the theorem holds for bridgeless planar graphs with two faces. Now let \mathcal{F} be the collection of the faces of G in this planar embedding and f_o be the outer face in \mathcal{F} . Choose an arbitrary face $f \neq f_o$ such that $E(f_o) \cap E(f) \neq \emptyset$. Set $G' = G \setminus (E(f_o) \cap E(f))$ and $\mathcal{F}' = \mathcal{F} \cup \{f_o \Delta f\} \setminus \{f_o, f\}$, where $f_o \Delta f$ is the face with edge set $E(f_o) \Delta E(f)$. Now, \mathcal{F}' is the collection of the faces of a planar embedding of G' . By the induction hypothesis, we can orient the edges of faces of G' such that \mathcal{F}' forms an OCDC, \mathcal{C}' . We denote the directed cycle of $f_o \Delta f$ by C_0 . Now we orient the edges of f_o and f such that the orientation of each edge of f_o and f is the same as its orientation in C_0 . These directed cycles of f_o and f are denoted by C_1 and C_2 , respectively. Therefore, $\mathcal{C} = \mathcal{C}' \cup \{C_1, C_2\} \setminus \{C_0\}$ is an OCDC of G . ■

Proposition 2.11 *Every bridgeless planar graph G with $|E(G)| < 2|V(G)| - 2$, has an SOCDC.*

Proof. Let G be a bridgeless planar graph. By the proof of Theorem 2.10, we can orient the edges of each face of G in such a way that the collection of the boundary of its faces, \mathcal{F} , is an OCDC. By Euler's formula, $|\mathcal{F}| = 2 + |E(G)| - |V(G)|$. Since $|E(G)| < 2|V(G)| - 2$, we conclude $|\mathcal{F}| < |V(G)|$. Hence, G has an SOCDC. ■

Since in every simple triangle-free planar graph G with at least three vertices, $|E(G)| \leq 2|V(G)| - 4$, we obtain the following corollary.

Corollary 2.12 *Let G be a bridgeless planar graph. If G is triangle-free, then G admits an SOCDC.*

The following proposition presents an SOCDC for the well-known non-planar triangle-free graphs.

Proposition 2.13 *Every $K_{n,m}$, $n, m \geq 2$, has an SOCDC.*

Proof. Assume that $V(K_{n,m}) = \{v_1, \dots, v_n; w_1, \dots, w_m\}$, $n \leq m$. Let

$$C_i = [v_1, w_i, v_2, w_{i+1}, v_3, w_{i+2}, \dots, v_{n-1}, w_{i+n-2}, v_n, w_{i+n-1}],$$

be a directed cycle, where subscripts are reduced modulo m . It is easy to check that $\mathcal{C} = \{C_i : 1 \leq i \leq m\}$ is an SOCDC of $K_{n,m}$, $n, m \geq 2$. ■

3 The minimal counterexample to the SOCDC conjecture

If the CDC conjecture is false, then it must have a minimal counterexample. It is proved that the minimal counterexample to this conjecture is a cubic, cyclically 4-edge-connected graph with edge chromatic number equal to 4 [7]. In this section, we study the properties of the minimal counterexample to the SOCDC conjecture.

Lemma 3.1 *If G is a graph with an SOCDC and G' is the graph obtained from G by dividing one edge of G , then G' also admits an SOCDC.*

Proof. Let G be a graph of order n with an SOCDC, say \mathcal{C} . Assume that G' is a graph obtained from G by dividing the edge $uv \in E(G)$ by a new vertex, say w . Let C_1 and C_2 be two directed cycles in \mathcal{C} that include the directed edges uv and vu , respectively. Define $C'_1 = C_1 \cup \{uw, vw\} \setminus \{uv\}$ and $C'_2 = C_2 \cup \{vw, wu\} \setminus \{vu\}$. Then $\mathcal{C}' = \mathcal{C} \cup \{C'_1, C'_2\} \setminus \{C_1, C_2\}$, is an OCDC of G' . Since $|\mathcal{C}'| = |\mathcal{C}| \leq n - 1$, \mathcal{C}' is an SOCDC of G' . ■

Theorem 3.2 [1] *Let $G = B_1 \cup \dots \cup B_k$ and B_i 's be blocks of G . If for each i , $1 \leq i \leq k$, B_i has an OPPDC or $B_i = K_3$ or K_5 , then G has an OPPDC.*

Theorem 3.3 *The minimal counterexample to the SOCDC conjecture is 3-regular.*

Proof. Let G be the minimal counterexample to the SOCDC conjecture. By Lemma 3.1, $\delta(G) \geq 3$. Let $v \in V(G)$ be a vertex of degree greater than 3. By H. Fleischner's vertex-splitting lemma [6], there exist two edges $e_1 = uv$ and $e_2 = vw \in E(G)$ such that $G' = G \cup \{uw\} \setminus \{e_1, e_2\}$ is bridgeless.

First assume that G' is 2-connected. Since $uv \notin E(G')$, $G' \neq K_4$ or K_6 . Therefore, G' has an SOCDC, say \mathcal{C}' . Let C'_1 and C'_2 be two directed cycles in \mathcal{C}' that include the directed edges uw and wu , respectively. Define $C_1 = C'_1 \cup \{uv, vw\} \setminus \{uw\}$ and $C_2 = C'_2 \cup \{wv, vu\} \setminus \{wu\}$. Then $\mathcal{C} = \mathcal{C}' \cup \{C_1, C_2\} \setminus \{C'_1, C'_2\}$, is an OCDC of G . Since $|\mathcal{C}| = |\mathcal{C}'|$, \mathcal{C} is an SOCDC of G' which is a contradiction.

Otherwise, $G' = G_1 \cup G_2$, where G_1 and G_2 are the blocks of G' and $V(G_1) \cap V(G_2) = \{s\}$. If every block of G' has an SOCDC, then by Proposition 2.1, G' has an SOCDC and we have done. Otherwise, at least one of its blocks is K_4 or K_6 . We consider the following cases and in each case we construct a smaller 2-connected graph and get to the contradiction as above.

(I) $G_1 = K_4$ and $G_2 \neq K_6$.

In this case, the following three possibilities may be happen.

If $u, w \in V(G_1)$ and $s = u$, then let $G'' = (G \setminus \{w, t\}) \cup \{vr\}$, where $V(G_1) = \{r, t, u, w\}$. Obviously, G'' is a 2-connected graph of order $|V(G)| - 2$. By the

minimality of G , G'' has an SOCDC, say \mathcal{C}'' , of size at most $|V(G)| - 3$. There exist two directed cycles $C_1, C_2 \in \mathcal{C}''$ such that C_1 and C_2 include the directed paths (v, r, u) and (u, r, v) , respectively. Now we define four new directed cycles. $C'_1 = (C_1 \setminus \{vr, ru\}) \cup \{vw, wt, tu\}$, $C'_2 = (C_2 \setminus \{rv\}) \cup \{rw, wv\}$, $C_3 = [r, u, t]$, and $C_4 = [t, w, r]$. Therefore, $\mathcal{C} = \mathcal{C}'' \cup \{C'_1, C'_2, C_3, C_4\} \setminus \{C_1, C_2\}$, is an OCDC of size at most $|V(G)| - 1$ of G , which is a contradiction.

If $u, w \in V(G_1)$ and $s \notin \{u, w\}$, then we change the role of v in G to s to obtain a smaller 2-connected graph. In the other word, $G'' = (G \setminus \{su, sw\}) \cup \{uw\}$ is a 2-connected graph where $|E(G'')| < |E(G)|$, thus we have done.

If $v \in V(G_1)$, then $s \neq v$, otherwise v is a cut vertex in G . Moreover, $s \notin \{u, w\}$, because uv and $wv \notin E(G')$. Now let $V(G_1) = \{r, s, t, v\}$. We change the role of w in G to t to obtain a smaller 2-connected graph. In the other word, $G'' = (G \setminus \{vt, vu\}) \cup \{tu\}$ is a 2-connected graph where $|E(G'')| < |E(G)|$, thus we have done.

(II) $G_1 = K_6$ and $G_2 \neq K_4$.

In this case, let $r \in V(G_1) \setminus \{u, v, w, s\}$. If $u, w \in V(G_1)$, then we change the role of v in G to r . For this purpose, let $G'' = (G \setminus \{ur, rw\}) \cup \{uw\}$. It is easy to check that G'' is a 2-connected graph where $|E(G'')| < |E(G)|$, thus we have done. Otherwise, $v \in V(G_1)$. Similar to Case (I) in above, we have $s \notin \{u, v, w\}$. Thus, we can change the role of w in G to r . In the other word, $G'' = (G \setminus \{uv, vr\}) \cup \{ur\}$ is a 2-connected graph where $|E(G'')| < |E(G)|$, thus we have done.

(III) Every block of G' is K_4 or K_6 .

In this case, since $e_i \notin E(G')$, $i = 1, 2$, $s \notin \{u, v, w\}$, moreover, the vertex v and the vertices u and w are in different blocks. By a discussion on the possible cases of the location of v , u and w in G_1 and G_2 , it can be seen that in every case $G \setminus s$ is the union of blocks which has an OPPDC. Thus by Theorem 3.2, $G \setminus s$ admits an OPPDC. Since s is adjacent to all vertex in G , by Theorem 1.3, G has an SOCDC, which is a contradiction.

Therefore, G is 3-regular. ■

Theorem 3.4 *The minimal counterexample to the SOCDC conjecture is 3-edge-connected.*

Proof. Let G be the minimal counterexample to the SOCDC conjecture and $F = \{e_1 = u_1v_1, e_2 = u_2v_2\}$ be an edge cut of size two in G . Since G is 2-connected, F is vertex disjoint. Let $G' = G.e_1$, the graph obtained from G by contracting the edge e_1 to the new vertex w . Since $G' \setminus w$ has the bridge $e_2 = u_2v_2$, $G' \neq K_4$ or K_6 . Therefore, G' has an SOCDC, say \mathcal{C}' . Let C'_1 and C'_2 be two directed cycles in \mathcal{C}' that include the directed edges u_2v_2 and v_2u_2 , respectively. Note that

$w \in V(C'_1) \cap V(C'_2)$. Define $C_1 = C'_1 \cup \{v_1u_1\} \setminus \{w\}$ and $C_2 = C'_2 \cup \{u_1v_1\} \setminus \{w\}$. Then $\mathcal{C} = \mathcal{C}' \cup \{C_1, C_2\} \setminus \{C'_1, C'_2\}$, is an OCDC of G . Since $|\mathcal{C}| = |\mathcal{C}'|$, \mathcal{C}' is an SOCDC of G' , which is a contradiction. ■

Since the minimal counterexample to the SOCDC, say G , is 3-regular, G is 3-edge-connected concludes that, G is also 3-connected.

A graph is called **cyclically k -edge-connected** if every edge cut separating the graph into non-acyclic components has at least k edges. An edge cut F , is called **trivial** if one of the component in $G \setminus F$ be an isolated vertex. One may check that a 3-connected cubic graph is cyclically 4-edge-connected if and only if it has no non-trivial edge cut of size 3.

Theorem 3.5 *The minimal counterexample to the SOCDC conjecture is cyclically 4-edge-connected.*

Proof. Assume that G is the minimal counterexample to the SOCDC conjecture. We know that G is 3-connected. Let G be not cyclically 4-edge-connected. It follows that $G = G_1 \cup G_2 \cup \{u_1v_1, u_2v_2, u_3v_3\}$, where $G_1 \cap G_2 = \emptyset$, the vertices u_i are distinct vertices of G_1 , and the vertices v_i are distinct vertices of G_2 , $i = 1, 2, 3$. Denote by H_i the graph obtained by contracting the subgraph G_{i+1} to a single vertex w_i , $i = 1, 2$, where subscripts are reduced modulo 2. Since $\deg(w_i) = 3$, $H_i \neq K_6$, $i = 1, 2$. By the minimality of G , H_i has an SOCDC or $H_i = K_4$. Therefore, H_i has an OCDC, \mathcal{C}_i , $i = 1, 2$. Let C_i^j , $j = 1, 2, 3$, be the three directed cycles in \mathcal{C}_i which include w_i , $i = 1, 2$, where without loss of generality, we assume that C_1^j includes directed path (u_{j-1}, w_1, u_{j+1}) , and C_2^j includes directed path (v_{j+1}, w_2, v_{j-1}) , where subscripts are reduced modulo 3, $j = 1, 2, 3$. Let $P_i^j = C_i^j \setminus w_i$, $i = 1, 2$, $j = 1, 2, 3$. Define $C^j = P_1^j \cup P_2^j \cup \{u_{j-1}v_{j-1}, v_{j+1}u_{j+1}\}$, $\mathcal{C}' = \{C^j : j = 1, 2, 3\}$, and $\mathcal{C}'' = \{C_i^j : i = 1, 2, j = 1, 2, 3\}$. Thus, $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}' \setminus \mathcal{C}''$ is an OCDC of G , where $|\mathcal{C}| = |\mathcal{C}_1| + |\mathcal{C}_2| - 3$. Note that every OCDC of K_4 has 4 cycles, therefore, in both cases $|\mathcal{C}| \leq |V(G)| - 1$, which is a contradiction. ■

Since the minimal counterexample to the SOCDC conjecture is cubic, the cyclically 4-edge connectivity of it conclude that its girth is at least 4.

Corollary 3.6 *The minimal counterexample to the SOCDC conjecture is triangle-free.*

Let G be a simple graph and (D, f) be an ordered pair where D is an orientation of $E(G)$ and f is a weight on $E(G)$ to \mathbb{Z} . For each $v \in V(G)$, denote

$$f^+(v) = \sum f(e) \quad \text{and} \quad f^-(v) = \sum f(e),$$

where the summation is taken over all directed edges of G (under the orientation D) with tails and heads, respectively, at the vertex v . An **integer flow** of G is an ordered pair (D, f) such that for every vertex $v \in V(G)$, $f^+(v) = f^-(v)$. A **nowhere-zero k -flow** of G is an integer flow (D, f) such that $0 < |f(e)| < k$, for every edge $e \in E(G)$ and is denoted by k -NZF [19].

By the following known results the edge chromatic number of the minimal counterexample to the SOCDC conjecture can be obtained.

Theorem 3.7 [19] *Every cubic graph G admits a 4-NZF if and only if $\chi'(G) = 3$.*

Theorem 3.8 [19] *A graph G admits a 4-NZF if and only if G has an OCDC consists of four directed even subgraphs.*

The following theorem concludes from Theorems 3.7 and 3.8.

Theorem 3.9 *Every cubic graph with edge chromatic number 3 admits an OCDC.*

Theorem 3.10 [10] *If \mathcal{C} is a CDC of a cubic graph G of order n , then $|\mathcal{C}| \leq n/2 + 2$.*

Since from every OCDC of a graph a CDC for the graph is obtained, we have the following corollary.

Corollary 3.11 *Every OCDC, \mathcal{C} , of a cubic graph of order $n \geq 6$, is an SOCDC.*

The following corollary concludes directly from Theorem 3.9 and Corollary 3.11.

Corollary 3.12 *Every cubic graph with edge chromatic number 3, $G \neq K_4$, has an SOCDC.*

Therefore the minimal counterexample to the SOCDC conjecture has the edge chromatic number equal to 4.

Corollary 3.13 *The minimal counterexample to the SOCDC conjecture is a snark.*

By Corollaries 3.11 and 3.13, the validity of the SOCDC conjecture is equivalent to the validity of the OCDC conjecture for snarks.

The properties of the minimal counterexample to the OPPDC conjecture is studied in [1, 13], and it is shown that this counterexample is a 2-connected and 3-edge-connected graph with minimum degree at least 4. Regarding to the relation between existence of an OPPDC for a graph and an SOCDC (Theorem 1.3), the

following relation between the order and the number of edges of these two minimal counterexamples can be obtained.

Assume that G_p and G_c are the minimal counterexamples to the OPPDC conjecture and the SOCDC conjecture, respectively. Let G'_p be a graph obtained from G_p by joining a new vertex to all vertex of G_p . By Theorem 1.3, G'_p has no SOCDC. Note that G_p is connected, so G'_p is 2-connected. Thus, G'_p is a counterexample to the SOCDC conjecture. Therefore by the minimality of G_c ,

$$|V(G_c)| + |E(G_c)| \leq |V(G'_p)| + |E(G'_p)| = 2|V(G_p)| + |E(G_p)| + 1.$$

Since G_c is cubic,

$$|V(G_c)| \leq \frac{2}{5}(2|V(G_p)| + |E(G_p)| + 1).$$

4 SOCDC and the Cartesian product

In [15] infinite classes of graphs with an SCDC are obtained using the Cartesian product of graphs. In this section, we proved the similar results in the oriented version.

The **Cartesian product** of two graphs G and H , denoted by $G \square H$, is the graph with vertex set $V(G) \times V(H)$ and two vertices (u, v) and (x, y) are adjacent if and only if either $u = x$ and $vy \in E(H)$ or $ux \in E(G)$ and $v = y$.

Consider the graph $G \square P_n$. For $v \in V(G)$ and $i \in V(P_n)$, we denote by v^i the vertex (v, i) of $G \square P_n$, and denote by G^i , $1 \leq i \leq n$, the subgraph of $G \square P_n$ induced by $\{v^i : v \in V(G)\}$; notice that G^i is simply a copy of G . If S is a directed path or directed cycle in G , then S^i denotes the corresponding directed path or directed cycle in G^i , S_* denotes a directed path or directed cycle in opposite direction of S , and if \mathcal{F} is a family of directed paths or directed cycles in G , then \mathcal{F}^i denotes the corresponding family in G^i .

Theorem 4.1 *If G has an OPPDC, then $G \square P_2$ has an SOCDC with $|V(G)|$ directed cycles. Furthermore, if G has an SOCDC, then $G \square P_n$, $n \geq 3$, has an SOCDC with at most $|V(G \square P_n)| - 1$ directed cycles.*

Proof. Let \mathcal{P} be an OPPDC of G , and suppose that $P \in \mathcal{P}$ begins from u to v . Then $P^1 \in \mathcal{P}^1$ begins from u^1 to v^1 , and $P^n \in \mathcal{P}^n$ begins from u^n to v^n . Thus,

$$C_P = P^1 \cup (v^1, v^2, \dots, v^{n-1}, v^n) \cup P_*^n \cup (u^n, u^{n-1}, \dots, u^2, u^1)$$

is a directed cycle in $G \square P_n$. Since \mathcal{P}^1 is an OPPDC of G^1 and \mathcal{P}^n is an OPPDC of G^n , the collection of directed cycles $\mathcal{C}_{\mathcal{P}} = \{C_P : P \in \mathcal{P}\}$ covers, twice in two

different directions, all edges of G^1 ; all edges of G^n ; and all edges between different copies of G . If $n = 2$, then $\mathcal{C}_{\mathcal{P}}$ is an SOCDC of $G \square P_2$ with $|V(G)|$ directed cycles. If $n \geq 3$, let \mathcal{C}^i be an SOCDC of G^i , $2 \leq i \leq n-1$. Then

$$\mathcal{C} = \mathcal{C}_{\mathcal{P}} \cup \left(\bigcup_{i=2}^{n-1} \mathcal{C}^i \right)$$

is an OCDC of $G \square P_n$, and

$$\begin{aligned} |\mathcal{C}| &= |\mathcal{C}_{\mathcal{P}}| + \sum_{i=2}^{n-1} |\mathcal{C}^i| \\ &\leq |V(G)| + \sum_{i=2}^{n-1} (|V(G)| - 1) \\ &= |V(G)| + (n-2)(|V(G)| - 1). \end{aligned}$$

Since $n \geq 3$, it follows that $|\mathcal{C}| \leq n|V(G)| - 1$, and thus \mathcal{C} is an SOCDC. \blacksquare

Now we need a theorem about the existence of OPPDC for the Cartesian product of graphs.

Theorem 4.2 [1] *If G and H have an OPPDC, then $G \square H$ also has an OPPDC.*

The following corollary follows directly from Theorems 4.2 and 4.1.

Corollary 4.3 *If G has an OPPDC, then for all $l \geq 2$, $G^l \square P_2$ has an SOCDC, where $G^l = \overbrace{G \square \cdots \square G}^{l \text{ times}}$.*

Corollary 4.4 *Every hypercube graph Q_n , $n \geq 2$, has an SOCDC.*

With the similar argument as above, the following theorems, which are the oriented version of some results in [15] for SCDC, can be proved.

Theorem 4.5 *If G has an OPPDC and an SOCDC, then for any tree, T , $G \square T$ has an SOCDC.*

Theorem 4.6 *If G has an OPPDC, then for all $k \geq 2$, $G \square C_{2k}$ has an SOCDC. Furthermore, if G has an SOCDC, then $G \square C_{2k-1}$ has an SOCDC.*

The following corollary concludes directly from Theorems 4.2 and 4.6.

Corollary 4.7 *If G has an OPPDC, then for all $k, l \geq 2$, $G^l \square C_{2k}$ has an SOCDC, where $G^l = \overbrace{G \square \cdots \square G}^{l \text{ times}}$.*

Theorem 4.8 *If G has an SOCDC, then for $n \geq 2|V(G)| + 1$, $G \square C_n$ has an SOCDC.*

Proof. Suppose that \mathcal{C} is an OCDC of size at most $|V(G)| - 1$ of G . Let \mathcal{D} be an OCDC of size 2 of C_n , \mathcal{C}^i , $1 \leq i \leq n$, be a copy of \mathcal{C} , which is an SOCDC of G^i , and \mathcal{D}^j , $1 \leq j \leq |V(G)|$, be a copy of \mathcal{D} , which is an SOCDC of C_n^j . Set

$$\mathcal{F} = \left(\bigcup_{i=1}^n \mathcal{C}^i \right) \cup \left(\bigcup_{j=1}^{|V(G)|} \mathcal{D}^j \right).$$

The collection \mathcal{F} is an OCDC of $G \square C_n$. Since $|\mathcal{C}| \leq |V(G)| - 1$, $|\mathcal{D}| = 2$, and $n \geq 2|V(G)| + 1$, we have $|\mathcal{F}| \leq n(|V(G)| - 1) + 2|V(G)| \leq n(|V(G)| - 1) + (n - 1) \leq n|V(G)| - 1 = |V(G \square C_n)| - 1$. Therefore, \mathcal{F} is an SOCDC of $G \square C_n$. ■

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